LOCAL AND POLLUTION ERROR ESTIMATION FOR STOKESIAN FLOWS

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SUMMARY

We describe in this paper an algebraic technique for estimating local and pollution errors in finite element approximations of Stokesian flows. © 1998 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The impact of adaptive methods on computational fluid dynamics is well recognized and can be regarded as one of the most important developments in CFD in several decades. The idea behind these approaches is to develop methods for error estimation (or error indication) and to use such estimates as a basis for adapting the mesh to reduce, control or equidistribute the numerical error. There is a large and still growing volume of evidence that such approaches can be highly effective and reduce substantially the number of unknowns needed in a given simulation to achieve a target error level; these approaches have even been successful in *p*- and *hp*-version finite element approximations.[1–5]

In recent times the success of adaptive schemes and *a posteriori* error estimation methods has prompted users to demand more information from such devices. In addition to giving rough estimates sufficient to deliver good meshes for controlling global approximations of energy or entropy, the need for determining local errors in energy norms or in other norms is persistently expressed. Moreover, there is naturally interest in determining errors in components of solutions of vector-valued functions, directional errors and elementwise errors in various norms.

The development of such desirable error estimators turns out to be a quite difficult task. Virtually all (there are exceptions) error estimators in use are global in structure. For example, the element

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residual method (ERM) of Oden *et al.*[6] and Ainsworth and Oden[7,8] yields bounds of the type

$$\|e\|_E \leqslant \left(\sum_{k=1}^{N_E} \eta_K^2 \right)^{1/2},$$

where $\|e\|_E$ is the global error in the energy norm and η_K is a local error indicator computed for element Ω_K in a mesh of N_E elements. The numbers η_K may be poor indicators of the actual errors in energy associated with Ω_K.

In recent work, Babuska *et al.*[9] and Oden and Feng[10] pointed out that techniques such as the ERM produce local estimates η_K *incapable of detecting 'pollution error'* produced by residuals outside the local region of interest. In order to obtain accurate local estimates, it was argued that this pollution error must also be estimated.

In the present paper we extend the work of Reference 10 to the case of steady Stokesian flows. The result is a new method for estimating local and pollution errors in finite element approximations of the Stokes problem. The approach is based on the idea that if $(\mathbf{u}_H, p_H)$ is the pair of velocity/pressure approximations calculated on a coarse mesh and $(\mathbf{u}_h, p_h)$ is a fine-mesh approximation, then $(\mathbf{u}_h - \mathbf{u}_H, p_h - p_H)$ is a reasonable approximation to the error. The fine-mesh approximation is never actually computed. By following the recent work of Bank and Smith[11] and Bank,[12] we are able to establish conditions sufficient to guarantee that such estimates are valid. An important side benefit is that the approach allows us to compute componentwise estimates, directional error estimates and local estimates in other norms.

2. PROBLEM SETTING

We consider steady, non-convecting, viscous flows characterized by the indefinite elliptic system embodied in the classical Stokes problem:

$$\text{find} \quad (\mathbf{u}, p) \in \mathbf{V} \times Q \quad \text{such that}$$
$$a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) = f(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \tag{1}$$
$$b(q, \mathbf{u}) = 0 \quad \forall q \in Q,$$

where

$$\mathbf{V} = \{\mathbf{v} \in (H^1(\Omega))^N : \mathbf{v} = \mathbf{0} \quad \text{on} \quad \partial\Omega\}, \qquad Q = \{q \in L^2(\Omega) : \int_\Omega q \, \mathrm{d}x = 0\},$$

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega v\nabla\mathbf{u} : \nabla\mathbf{v} \, \mathrm{d}x, \qquad b(q, \mathbf{v}) := \int_\Omega -\mathrm{div}\,\mathbf{v} q \, \mathrm{d}x,$$

with Ω a smooth bounded domain in $\mathbb{R}^N$, $N = 2$ or 3.

As usual, Ω is decomposed into a family of partitions $\mathscr{P}_h$ of finite element meshes over which general *hp*-approximations of the field $\mathbf{u}$ and p are constructed. The family is assumed to be regular so that standard local interpolation estimates hold.[10] Let $\mathscr{P}_{h=H}$ be a particular 'coarse' mesh on which we approximate (1) on subspaces $\mathbf{V}^H \subset \mathbf{V}$ and $Q^H \subset Q$ and let $\mathbf{V}^h$ and Q^h denote finer meshes in a partition $\mathscr{P}_h \supset \mathscr{P}_H$. We write

$$\mathbf{V}^h = \mathbf{V}^H + \mathbf{V}^{hH} \subset \mathbf{V}, \qquad Q^h = Q^H + Q^{hH} \subset Q. \tag{2}$$

The approximations to (1) in $(\mathbf{V}^H, Q^H)$ and $(\mathbf{V}^h, Q^h)$ are then

$$(\mathbf{u}^H, p^H) \in \mathbf{V}^H \times Q^H \quad \text{such that} \quad \forall(\mathbf{v}^H, q^H) \in \mathbf{V}^H \times Q^H$$
$$a(\mathbf{u}^H, \mathbf{v}^H) + b(p^H, \mathbf{v}^H) - b(q^H, \mathbf{u}^H) = f(\mathbf{v}^H) \tag{3}$$

and

$$(\mathbf{u}^h, p^h) \in \mathbf{V}^h \times Q^h \quad \text{such that} \quad \forall (\mathbf{v}^h, q^h) \in \mathbf{V}^h \times Q^h$$
$$a(\mathbf{u}^h, \mathbf{v}^h) + b(p^h, \mathbf{v}^h) - b(q^h, \mathbf{u}^h) = f(\mathbf{v}^h). \tag{4}$$

We can write down sufficient conditions for $(\mathbf{u}^h - \mathbf{u}^H, p^h - p^H)$ to be a reasonable approximation of $(\mathbf{u} - \mathbf{u}^H, p - p^H)$ and we summarize these in Section 4. With the assumption that such conditions do indeed hold, we focus on properties of the errors

$$\mathbf{e}^H = \mathbf{u}^h - \mathbf{u}^H, \qquad E^H = p^h - p^H. \tag{5}$$

3. STRUCTURE OF THE ERROR

Introducing (5) into (4) gives

$$a(\mathbf{e}^H, \mathbf{v}^h) + b(E^H, \mathbf{v}^h) - b(q^h, \mathbf{e}^H) = R(\mathbf{v}^h, q^h) \quad \forall (\mathbf{v}^h, q^h) \in \mathbf{V}^h \times Q^h, \tag{6}$$

where $R(\mathbf{v}^h, q^h)$ is the fine-mesh residual given by

$$R(\mathbf{v}^h, q^h) = f(\mathbf{v}^h) - a(\mathbf{u}^H, \mathbf{v}^h) - b(p^H, \mathbf{v}^h) + b(q^h, \mathbf{u}^H). \tag{7}$$

Each error component can be expressed as a linear combination of the basis functions spanning $\mathbf{V}^h$, $\mathbf{V}^{hH}$, Q^h and Q^{hH}; symbolically,

$$\mathbf{e}^H(\mathbf{x}) = \sum_{m=1}^{N_h} \mathscr{E}_{\mathbf{u}}^m \chi_m(\mathbf{x}), \qquad E^H(\mathbf{x}) = \sum_{k=1}^{M_h} \mathscr{E}_p^k \psi_k(\mathbf{x}), \tag{8}$$

where $\{\chi_m\}$ and $\{\psi_k\}$ are appropriate fine-mesh bases. Thus introducing (8) into (6) leads to a linear algebraic system for the error coefficients $\mathscr{E}_{\mathbf{u}}^m$ and $\mathscr{E}_p^k$:

$$\mathbf{A}\mathscr{E} = \mathbf{R}, \tag{9}$$

$\mathbf{A}$ being the global fine-mesh stiffness matrix and $\mathbf{R}$ being the residual vector.

Let $\Omega_K \in \mathscr{P}_H$ be a typical element (or patch) in the coarse mesh, Γ_I be its interface with the remaining elements and $\Omega_{\tilde{K}}$ be the remaining elements:

$$\bar{\Omega}_{\tilde{K}} = \bigcup_{L=1}^{N_E} \bar{\Omega}_L \backslash (\Omega_K \cup \Gamma_I).$$

Then (9) can be written as

$$\begin{bmatrix} \mathbf{A}_K & \mathbf{A}_{KI} & \mathbf{0} \\ \mathbf{A}_{IK} & \mathbf{A}_I & \mathbf{A}_{I\tilde{K}} \\ \mathbf{0} & \mathbf{A}_{\tilde{K}I} & \mathbf{A}_{\tilde{K}} \end{bmatrix} \begin{bmatrix} \mathscr{E}_K \\ \mathscr{E}_I \\ \mathscr{E}_{\tilde{K}} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_K \\ \mathbf{R}_I \\ \mathbf{R}_{\tilde{K}} \end{bmatrix} = \mathbf{R}. \tag{10}$$

We decompose $\mathbf{R}$ as

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_K \\ \mathbf{R}_I \\ \mathbf{R}_{\tilde{K}} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_K \\ \mathbf{R}_I^{(K)} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{R}_I^{(\tilde{K})} \\ \mathbf{R}_{\tilde{K}} \end{bmatrix}, \tag{11}$$

so that

$$\begin{bmatrix} \mathbf{A}_K & \mathbf{A}_{KI} & \mathbf{0} \\ \mathbf{A}_{IK} & \mathbf{A}_I & \mathbf{A}_{I\tilde{K}} \\ \mathbf{0} & \mathbf{A}_{\tilde{K}I} & \mathbf{A}_{\tilde{K}} \end{bmatrix} \begin{bmatrix} \mathscr{E}_K^{\text{loc}} \\ \mathscr{E}_I^{(K)} \\ \mathscr{E}_{\tilde{K}}^{\text{pol}} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_K \\ \mathbf{R}_I^{(K)} \\ \mathbf{0} \end{bmatrix} \tag{12}$$

and

$$\begin{bmatrix} \mathbf{A}_K & \mathbf{A}_{KI} & \mathbf{0} \\ \mathbf{A}_{IK} & \mathbf{A}_I & \mathbf{A}_{I\tilde{K}} \\ \mathbf{0} & \mathbf{A}_{\tilde{K}I} & \mathbf{A}_{\tilde{K}} \end{bmatrix} \begin{bmatrix} \mathscr{E}_K^{\text{pol}} \\ \mathscr{E}_I^{(\tilde{K})} \\ \mathscr{E}_{\tilde{K}}^{\text{loc}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{R}_I^{(\tilde{K})} \\ \mathbf{R}_{\tilde{K}} \end{bmatrix}. \tag{13}$$

The solutions of (12) and (13) provide the complete relative representation of errors on Ω and reveal how the residuals on $\Omega_{\tilde{K}}$ affect the errors on Ω_K and *vice versa*. The local error component $\mathscr{E}_K^{\text{loc}}$ is due to the residuals on the interior degrees of freedom on Ω_K, the error component $\mathscr{E}_K^{\text{pol}}$ is the pollution error over $\Omega_{\tilde{K}}$ from the residuals on Ω_K, while the error component $\mathscr{E}_I^{(K)}$ is the local error when it is associated with Ω_K and is the pollution error when it is associated with $\Omega_{\tilde{K}}$. Analogous interpretations apply to pollution and local errors on $\Omega_{\tilde{K}}$. Thus the local and pollution error components on Ω_K can be written in the form

$$\bar{\mathscr{E}}_K^{\text{loc}} = \begin{bmatrix} \mathscr{E}_K^{\text{loc}} \\ \mathscr{E}_I^{(K)} \end{bmatrix}, \qquad \bar{\mathscr{E}}_K^{\text{pol}} = \begin{bmatrix} \mathscr{E}_K^{\text{pol}} \\ \mathscr{E}_I^{(K)} \end{bmatrix} \tag{14}$$

and the total error on Ω_K (including its boundary Γ_I) is

$$\bar{\mathscr{E}}_K = \bar{\mathscr{E}}_K^{\text{loc}} + \bar{\mathscr{E}}_K^{\text{pol}}. \tag{15}$$

It is important to note that the matrix $\mathbf{A}_I^{-1}$ does not need to be computed explicitly. A procedure to handle this is described shortly.

A simple calculation reveals that[10]

$$\bar{\mathscr{E}}_K^{\text{loc}} = \begin{bmatrix} \mathbf{A}_K & \mathbf{A}_{KI} \\ \mathbf{A}_{IK} & \tilde{\mathbf{A}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{R}_K \\ \mathbf{R}_I^{(K)} \end{bmatrix}, \qquad \bar{\mathscr{E}}_K^{\text{pol}} = \begin{bmatrix} -\mathbf{A}_K^{-1}\mathbf{A}_{KI} \\ \mathbf{I} \end{bmatrix} \bar{\mathscr{E}}_I^{(\tilde{K})}, \tag{16}$$

where

$$\tilde{\mathbf{A}} = \mathbf{A}_I - \mathbf{A}_{I\tilde{K}}\mathbf{A}_{\tilde{K}}^{-1}\mathbf{A}_{\tilde{K}I}, \qquad \bar{\mathscr{E}}_I^{(\tilde{K})} = \tilde{\mathbf{A}}^{-1}(\mathbf{R}_I^{(K)} - \mathbf{A}_{I\tilde{K}}\mathscr{E}_{\tilde{K}}^{\text{loc}}). \tag{17}$$

Introducing these into (8) determines the pointwise errors.

A simple and efficient algorithm can be constructed to solve these equations assuming that the coarse-mesh solution is known. First we observe the following.

1. The coarse-mesh solution $(\mathbf{u}_H, p_H)$ and the associated stiffness matrix $\mathbf{A}_H$ are assumed to be known and available for use in error estimation and $\mathbf{A}_H$ is readily factored into upper and lower triangular matrices: $\mathbf{A}_H = \mathbf{L}_H\mathbf{U}_H$.
2. Globally, if $\mathbf{A}_h$ is the fine-mesh stiffness matrix, then for any element Ω_K (symbolically, with vector notation displaced momentarily)

$$\mathbf{A}_h\mathscr{E}_K^{\text{loc}} + \mathbf{A}_h\mathscr{E}_K^{\text{pol}} = \mathbf{R}_K + \mathbf{R}_K^{h,\text{pol}}$$

and

$$\mathscr{E}_K^{\text{pol}} = \mathbf{A}_h^{-1}\mathbf{R}_K^{h,\text{pol}}.$$

3. A readily calculable coarse-mesh approximation of $\mathscr{E}_K^{\text{pol}}$ is then

$$\mathbf{A}_H^{-1}\mathbf{R}_K^{H,\text{pol}},$$

where $\mathbf{R}_K^{H,\text{pol}}$ is the submatrix of $\mathbf{R}_K^{h,\text{pol}}$ corresponding to the coarse-mesh degrees of freedom.
4. The local problem for the ERM local error indicator $\varphi_K^{h,\text{pol}}$ yields

$$\mathbf{A}_K\varphi_K^{h,\text{pol}} = \mathbf{A}_K(\mathbf{A}_H^{-1}\mathbf{R}_K^{H,\text{pol}})_{\Omega_K} \approx \mathbf{R}_K^{\text{pol}}, \quad 1 \leqslant K \leqslant N_E, \tag{18}$$

which then yields a local pollution error indicator.

The calculation of $\mathbf{A}_K^{-1}\mathbf{R}_K^{H,\text{pol}}$ involves only a back substitution using $\mathbf{U}_H$. Moreover, the calculation of $\mathbf{R}_K^{h,\text{pol}}$ can be accomplished using a D/SAXPY operation on the ERM error indicators. The cost of the entire process beginning with a knowledge of $\mathbf{A}_H$ and the coarse-mesh residuals is $\mathscr{O}(N_H^2)$ and this can be reduced to $\mathscr{O}(N_H \log N_H)$ by fast matrix–vector multiplication algorithms. The full algorithm is as follows.

Step 1. Solve the coarse-mesh problem for $\mathbf{u}^H, p^H \in \mathbf{V}^H \times Q^H$.
Step 2. Construct the perturbation space $\mathbf{V}^{hH}$ and compute residuals.
Step 3. Compute local error indicator φ_K^h ($1 \leqslant K \leqslant N_E$) using the ERM.[10]
Step 4. Calculate global pollution residuals from element $\Omega_J, J \neq K$.
Step 5. Calculate equivalent pollution residual $\mathbf{R}_K^{\text{pol}}$ on Ω_K based on $\mathbf{R}_K^{h,\text{pol}}$, $\mathbf{R}_K^{h,\text{pol}} \to \mathbf{R}_K^{\text{pol}}$.
    (a) Compute $\mathbf{R}_K^{H,\text{pol}}$ from $\mathbf{R}_K^{h,\text{pol}}$ via a Schur complement type of operation.
    (b) Compute pollution residual $\mathbf{R}_K^{\text{pol}} := \mathbf{A}_K(\mathbf{A}_H^{-1}\mathbf{R}_K^{H,\text{pol}})_{\Omega_K}$ on Ω_K.
Step 6. Solve for pollution estimate $\varphi_K^{h,\text{pol}}$ using (18).

Up to Step 3, no more calculations have been done than are ordinarily required to compute error indicators in conventional schemes based on the ERM. As noted earlier, the matrix operation equivalent to Step 4 is a D/SAXPY operation, which can be executed efficiently using BLAS library routines. Various parallel computing techniques can be adopted in this step.

4. SUFFICIENT CONDITIONS FOR USE OF FINE-MESH APPROXIMATIONS

We introduce sufficient conditions for $(\mathbf{u}^h - \mathbf{u}^H, p^h - p^H)$ to be a reasonable approximation of $(\mathbf{u} - \mathbf{u}^H, p - p^H)$. Following References 11 and 12, we assume the solutions $(\mathbf{u}, p)$, $(\mathbf{u}^H, p^H)$ and $(\mathbf{u}^h, p^h)$ of the respective problems (1), (3) and (4) to fulfil the *saturation assumption*, i.e. there exists a positive constant $\sigma < 1$ such that

$$\|(\mathbf{u}, p) - (\mathbf{u}^h, p^h)\| \leqslant \sigma\|(\mathbf{u}, p) - (\mathbf{u}^H, p^H)\|, \tag{19}$$

where $\|\cdot\|$ is a given norm on the product space $\mathbf{V} \times Q$.
For convenience we define the bilinear form $\mathscr{A} \colon (\mathbf{V} \times Q) \times (\mathbf{V} \times Q) \to \mathbb{R}$:

$$\mathscr{A}((\mathbf{u}, p), (\mathbf{v}, p)) = a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) - b(q, \mathbf{u}). \tag{20}$$

The bilinear form $\mathscr{A}$ is continuous and satisfies the standard inf–sup condition with respect to the norm $\|\cdot\|$, conditions required for solvability of problem (1). We also assume that $\mathscr{A}$ satisfies a discrete inf–sup condition. In particular, there exist positive constants M and $\alpha > 0$ such that for all $(\mathbf{u}, p), (\mathbf{v}, q) \in \mathbf{V} \times Q$

$$\mathscr{A}((\mathbf{u}, p), (\mathbf{v}, q)) \leqslant M\|(\mathbf{u}, p)\|\|(\mathbf{v}, q)\|$$

and for every $(\mathbf{u}^h, p^h) \in \mathbf{V}^h \times Q^h$

$$\sup_{\|(\mathbf{v}^h, q^h)\| \leqslant 1} \frac{|\mathscr{A}((\mathbf{u}^h, p^h), (\mathbf{v}^h, q^h))\|}{\|(\mathbf{v}^h, q^h)\|} \geq \alpha \|(\mathbf{u}^h, p^h)\|.$$

We are now able to show that $(\mathbf{u}^h - \mathbf{u}^H, p^h - p^H)$ forms a reasonable approximation of $(\mathbf{u} - \mathbf{u}^H, p - p^H)$ in the sense that there exist two positive constants C_1 and C_2 such that

$$C_1 \|(\mathbf{u} - \mathbf{u}^H, p - p^H)\| \leqslant \|(\mathbf{u}^h - \mathbf{u}^H, p^h - p^H)\| \leqslant C_2 \|(\mathbf{u} - \mathbf{u}^H, p - p^H)\|. \tag{21}$$

The derivation of the lower bound immediately follows from the *saturation assumption*. Indeed,

$$\begin{aligned}
\|(\mathbf{u} - \mathbf{u}^H, p - p^H)\| &\leqslant \|(\mathbf{u} - \mathbf{u}^h, p - p^h)\| + \|(\mathbf{u}^h - \mathbf{u}^H, p^h - p^H)\| \\
&\leqslant \sigma \|(\mathbf{u} - \mathbf{u}^H, p - p^H)\| + \|(\mathbf{u}^h - \mathbf{u}^H, p^h - p^H)\|,
\end{aligned}$$

so that

$$(1 - \sigma) \|(\mathbf{u} - \mathbf{u}^H, p - p^H)\| \leqslant \|(\mathbf{u}^h - \mathbf{u}^H, p^h - p^H)\|. \tag{22}$$

In order to derive the upper bound, we first observe from (1) and (4) that

$$\mathscr{A}((\mathbf{u}, p) - (\mathbf{u}^h, p^h), (\mathbf{v}^h, q^h)) = 0 \quad \forall (\mathbf{v}^h, q^h) \in \mathbf{v}^h \times Q^h, \tag{23}$$

which yields

$$\mathscr{A}((\mathbf{u}, p) - (\mathbf{v}^h, q^h)) = \mathscr{A}((\mathbf{u}^h, p^h), (\mathbf{v}^h, q^h)) \quad \forall (\mathbf{v}^h, q^h) \in \mathbf{v}^h \times Q^h. \tag{24}$$

Then, because $\mathbf{V}^H \subset \mathbf{V}^h$ and $Q^H \subset Q^h$ and from the inf–sup condition in the product space $\mathbf{V}^h \times Q^h$, we have

$$\begin{aligned}
\alpha \|(\mathbf{u}^h, p^h) - (\mathbf{u}^H, p^H)\| &\leqslant \sup_{\|(\mathbf{v}^h, q^h)\| \leqslant 1} \frac{|\mathscr{A}((\mathbf{u}^h, p^h) - (\mathbf{u}^H, p^H), (\mathbf{v}^h, q^h))|}{\|(\mathbf{v}^h, q^h)\|} \\
&\leqslant \sup_{\|(\mathbf{v}^h, q^h)\| \leqslant 1} \frac{|\mathscr{A}((\mathbf{u}, \mathbf{p}) - (\mathbf{u^H}, \mathbf{p^H}), (\mathbf{v^h}, \mathbf{q^h}))|}{\|(\mathbf{v}^h, q^h)\|} \\
&\leqslant \sup_{\|(\mathbf{v}^h, q^h)\| \leqslant 1} \frac{M \|(\mathbf{u}, \mathbf{p}) - (\mathbf{u^H}, \mathbf{p^H})\| \|(\mathbf{v^h}, \mathbf{q^h})\|}{\|(\mathbf{v}^h, q^h)\|} \leqslant M \|(\mathbf{u}, p) - (\mathbf{u}^H, p^H)\|,
\end{aligned}$$

which yields the upper bound

$$\|(\mathbf{u}^h, p^h) - (\mathbf{u}^H, p^H)\| \leqslant \frac{M}{\alpha} \|(\mathbf{u}, p) - (\mathbf{u}^H, p^H)\|. \tag{25}$$

5. NUMERICAL EXPERIMENTS

We present in this section error estimation results for the Stokes problem. We define a two-dimensional model problem on a unit square domain $(0, 1) \times (0, 1)$ for which the solution $(\mathbf{u}, p)$ is smooth and given by

$$u = 5x^4 y - 10x^2 y^3 + y^5, \qquad v = x^5 - 10x^3 y^2 + 5xy^4 + 5x^2, \qquad p = y - 3.$$

The numerical results are presented for the finite element solution obtained on the final adapted *hp*-mesh. First a solution is computed on a uniform coarse mesh, which is then refined in h and further enriched in p in order to reduce the error to a given target error. The adaptation is performed based on the error estimators we have developed. In Plates 1 and 2 respectively we show the ERM error

estimates and the pollution error estimates computed according to the algorithm presented in Section 3. Since we know the exact solution, we also calculate the effectivity index as the ratio between the computed error and the exact error. The global effectivity index for the error estimates with pollution correction has a value of 1·052. We also provide the effectivity indices for each element, displayed in Plate 3, which range from 0·997 to 1·12.

Plates 4 and 5 respectively are plots of the exact error in the first velocity component and the corresponding elementwise effectivity indices. This time they range from 0·918 to 2·14, but the global effectivity index is about 1·25. On the other hand, the elementwise effectivity indices for pressure, shown in Plate 7, vary from 1·00 to 1·24 only; this allows us to conclude that the error estimates for pressure give a very good approximation of the exact error, which is shown in Plate 6.

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